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NON-LOCAL AND LOCAL ARTIFICIAL BOUNDARY CONDITIONS FOR TWO-DIMENSIONAL FLOW IN AN INFINITE CHANNEL

IGOR PATLASHENKO AND DAN GIVOLI

Asher Space Research Center, Department of Aerospace Engineering, Technion-Israel Institute of Technology, Haifa 32000, Israel

ABSTRACT

The numerical solution of problems involving two-dimensional flow in an infinite or a semi-infinite channel is considered. Beyond a certain finite region, where the flow and geometry may be general, a "tail" region is assumed where the flow is potential and the channel is uniform. This situation is typical in many cases of fluid-structure interaction and flow around obstacles in a channel. The unbounded domain is truncated by means of an artificial boundary B, which separates between the finite computational domain and the "tail." On B, special boundary conditions are devised. In the finite computational domain, the problem is solved using a finite element scheme. Both non-local and local artificial boundary conditions are considered on B, and their performance is compared via numerical examples

KEY WORDS Artificial boundary conditions Finite elements Infinite channel

INTRODUCTION

Fluid domains which extend to infinity are encountered frequently in problems involving fluid-structure interaction and flow around obstacles. There are several numerical methods which deal with infinite domains[1]. One commonly used method consists of the truncation of the unbounded domain by means of an artificial boundary B, leading to the definition of a finite computational domain Ω. On B, a specially-devised boundary condition is imposed. Then the problem in Ω is solved using the finite element method or some other numerical method.

A standard boundary condition which is often imposed on B is simply the condition at infinity (i.e., uniform flow). However, in this case Ω must be quite large, or else the artificial boundary condition would give rise to spurious reflections and would pollute the numerical solution[2]. On the other hand, a large computational domain is inefficient, leading to a large number of degrees of freedom. Therefore, the trend in recent works is to use a more accurate boundary condition on B. During the last two decases, various special artificial boundary conditions for flows in infinite domains have been proposed[3–9] which enable the use of a smaller computational domain and thus reduce the computing effort.

Problems of flows in channels, ducts and wave guides lead to mathematical models involving an infinite or a semi-infinite strip. One interesting case is that of water waves of seismic origin near a dam[10,11]. Usually, beyond some finite region Ω, a "tail" region D is defined where the strip is assumed to be uniform and the governing equations are assumed to be linear and homogeneous (i.e. no sources are present). On the other hand, in Ω the geometry and governing equations may be more general, limited only by the capabilities of the finite element scheme employed. For example, Tsynkov and co-workers[12,13] consider, for two-dimensional stationary

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exterior problems, the compressible Euler equations in Ω and the equations of linear potential flow in D[12], or the compressible Navier Stokes equations in Ω and their linearized version in D[13]. The artificial boundary B separates the computational domain Ω from the tail D. The boundary conditions used on B must effectively represent the behavior of the solution in D[14–17]. The large majority of the artificial boundary conditions previously proposed are local and approximate.

In this paper an exact non-local boundary condition is devised on an artificial boundary B, for problems involving two-dimensional flow in an infinite or a semi-infinite channel. The flow in the tail region D is assumed to be potential, although no such restriction need to be made in the computational domain Ω. The method for deriving the exact non-local condition is similar to that used by Keller and Givoli[18–20] for exterior problems. Then, the exact non-local condition is *localised* and a sequence of local approximate boundary conditions is obtained. These local conditions are exact for solutions consisting of a finite number of harmonics. For high-order local boundary conditions, the conforming finite elements devised by Givoli and Keller[21] are used in the layer near B. The performance of the non-local and local conditions is compared via numerical experiments.

STATEMENT OF THE PROBLEM

Consider the two-dimensional flow in a semi-infinite channel, as shown in *Figure 1*. In the domain $0 \leqslant x \leqslant x_0$, which is denoted Ω, the geometry may be general. On the other hand, in the tail region $x \geqslant x_0$, denoted D, the y coordinate ranges from 0 to a constant width b. The upper and lower boundaries of D are denoted Γ_U and Γ_L, respectively. The interface $x = x_0$ between Ω and D is denoted B. The rest of the boundary of Ω, namely $\partial\Omega - B$, is denoted S. Although the channel considered here is semi-infinite, the procedures described in what follows may be applied to a two-sided infinite channel as well.

Some equations govern in Ω, representing some type of fluid flow, depending on the problem at hand and on the capabilities of the finite element scheme to be employed in Ω. Some appropriate boundary conditions are given on S and on the boundaries of obstacles in Ω, if any. On the other hand, in the tail region D, it is assumed that the flow is potential, so that Laplace's equation governs,

$$\nabla^2 u = 0 \qquad \text{in } D \tag{1}$$

Here u is either the velocity potential or the stream function. As to the boundary conditions on Γ_U and Γ_L and the boundary condition at infinity, two cases are considered,

Case 1:

$$\frac{\partial u}{\partial y} = 0 \qquad \text{on } \Gamma_U, \Gamma_L \tag{2}$$

$$\frac{\partial u}{\partial x} \to V_\infty \qquad \text{as } x \to \infty \tag{3}$$

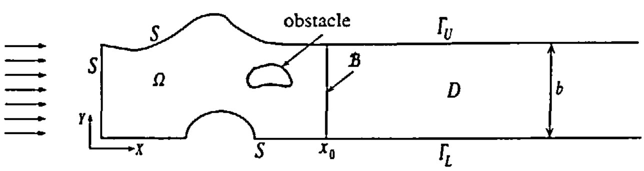

Figure 1 A semi-infinite channel with a uniform tail

Case 2:

$$u=0 \quad \text{on } \Gamma_U, \Gamma_L \tag{4}$$

$$u \to 0 \quad \text{as } x \to \infty \tag{5}$$

In *Case 1*, u is the velocity potential, (2) is a non-penetration boundary condition, and (3) implies that the fluid has uniform velocity V_∞ in the x direction at infinity. *Case 2* has a less direct interpretation (it may be obtained as an incremental problem when u is the stream function), but it will turn out to be simpler than *Case 1* in some aspects. More importantly, the two cases are basic prototypes for more general problems. In both cases, it is assumed that the differential equation given in Ω and the boundary conditions given on S are such that the solution to the problem exists and is unique.

EXACT BOUNDARY CONDITIONS ON AN ARTIFICIAL BOUNDARY

The method of solution adopted here is based on replacing the original problem in $\Omega \cup D$ by a new problem in Ω alone. This is done by imposing an appropriate boundary condition on B, thus completing the statement of the problem in Ω. Then the new problem in Ω is solved by the finite element method. In this section, an *exact* non-local boundary condition is devised on B. With the use of this exact condition, the new problem in Ω is *equivalent* to the original problem in $\Omega \cup D$.

Case 1 is considered first. In this case, u satisfied (1)–(3). The general solution to these equations is found, by separation of variables, to be,

$$u(x, y) = V_\infty x + \sum_{n=0}^{\infty} U_n \cos(n\pi y/b) \exp(-n\pi x/b) \quad \text{in } D, \tag{6}$$

where,

$$U_0 = \frac{1}{b} \int_0^b u(x_0, y)\,\mathrm{d}y - V_\infty x_0 \tag{7}$$

$$U_n = \frac{2}{b} \exp(n\pi x_0/b) \int_0^b \cos(n\pi y/b) u(x_0, y)\,\mathrm{d}y, \qquad n \geq 1. \tag{8}$$

By differentiating (6) with respect to x and setting $x = x_0$, one obtains,

$$\frac{\partial u}{\partial x}(x_0, y) = V_\infty - \sum_{n=1}^{\infty} \frac{2n\pi}{b^2} \int_0^b \cos(n\pi y/b) \cos(n\pi y'/b) u(x_0, y')\,\mathrm{d}y'. \tag{9}$$

Equation (9) is an exact non-local boundary condition on B. As in the case of exterior problems[18,19], it is called a Dirichlet-to-Neumann (DtN) boundary condition, because it relates the Dirichlet datum u and the Neumann datum $\partial u/\partial x$ on B.

Now *Case 2* is considered. In this case, u satisfies (1), (4) and (5). Analogously to (6)–(9), one obtains,

$$u(x, y) = \sum_{n=1}^{\infty} U_n \sin(n\pi y/b) \exp(-n\pi x/b) \qquad \text{in } D, \tag{10}$$

$$U_n = \frac{2}{b} \exp(n\pi x_0/b) \int_0^b \sin(n\pi y/b) u(x_0, y)\,\mathrm{d}y, \tag{11}$$

$$\frac{\partial u}{\partial x}(x_0, y) = -\sum_{n=1}^{\infty} \frac{2n\pi}{b^2} \int_0^b \sin(n\pi y/b) \sin(n\pi y'/b) u(x_0, y')\,\mathrm{d}y'. \tag{12}$$

Equation (12) is an exact non-local DtN boundary condition on B.

LOCALIZED BOUNDARY CONDITIONS: CASE 2

The non-local DtN boundary conditions (9) and (12) are exact. However, in practice the infinite sum in (9) and in (12) is truncated after a finite number of terms, N. Of course, the truncated boundary condition is not exact in general. It is exact only if the solution u consists of (at most) the first N harmonics. The goal of this and the next section is to develop *local* boundary conditions which also have the same property. Local boundary conditions have the advantage over non-local ones in that they lead to finite element schemes with standard architecture[21].

Case 2 is considered first because it turns out to be simpler. Consider solutions u consisting of the first N harmonics. Then u on B has the form,

$$u(x_0, y) = \sum_{n=1}^{N} A_n \sin(n\pi y/b). \tag{13}$$

Here the A_n are constants (Fourier coefficients). By substituting (13) in (12) and making use of the orthogonality of the sine functions, one gets,

$$\frac{\partial u}{\partial x}(x_0, y) = -\frac{\pi}{b} \sum_{n=1}^{\infty} A_n n \sin(n\pi y/b). \tag{14}$$

Now, it is desired to find a linear differential operator L_N, *which does not depend on n*, such that

$$n \sin(n\pi y/b) = L_N[\sin(n\pi y/b)], \qquad n = 1, \ldots, N. \tag{15}$$

With such an operator at hand, (14) gives,

$$\frac{\partial u}{\partial x}(x_0, y) = -\frac{\pi}{b} L_N\left[\sum_{n=1}^{\infty} A_n \sin(n\pi y/b)\right], \tag{16}$$

and by using (13) one finally obtains,

$$\frac{\partial u}{\partial x}(x_0, y) = -\frac{\pi}{b} L_N[u(x_0, y)]. \tag{17}$$

Equation (17) is a local boundary condition on B which is exact for all solutions consisting of up to the first N harmonics.

It remains to find an operator L_N with the property (15). To this end n is first written as a finite sum of polynomials of n^2, namely,

$$n = \sum_{m=0}^{N-1} \alpha_m^{(N)} P_m(n^2); \qquad n = 1, \ldots, N. \tag{18}$$

Here $P_m(k)$ is a polynomial of degree m in k, and the $\alpha_m^{(N)}$ are constants. For given polynomials P_m, (18) is a system of N linear algebraic equations for the N coefficients $\alpha_m^{(N)}$. Next, it is easy to show that,

$$P_m(n^2) \sin(n\pi y/b) = P_m\left(-(b/\pi)^2 \frac{\partial^2}{\partial y^2}\right) \sin(n\pi y/b). \tag{19}$$

To prove this, the polynomial P_m is written explicitly as $P_m(k) = \sum_{j=0}^{m} \beta_j k^j$. Then,

$$\begin{aligned} P_m(n^2) \sin(n\pi y/b) &= \sum_{j=0}^{m} \beta_j n^{2j} \sin(n\pi y/b) = \sum_{j=0}^{m} \beta_j (-1)^j (b/\pi)^{2j} \frac{\partial^{2j}}{\partial y^{2j}} \sin(n\pi y/b) \\ &= P_m\left(-(b/\pi)^2 \frac{\partial^2}{\partial y^2}\right) \sin(n\pi y/b), \end{aligned} \tag{20}$$

which yields (19). Now, the operator L_N is defined by,

$$L_N = \sum_{m=0}^{N-1} \alpha_m^{(N)} P_m\left(-(b/\pi)^2 \frac{\partial^2}{\partial y^2}\right). \tag{21}$$

Then (15) indeed holds, since,

$$n \sin(n\pi y/b) = \sum_{m=0}^{N-1} \alpha_m^{(N)} P_m(n^2) \sin(n\pi y/b) = \sum_{m=0}^{N-1} \alpha_m^{(N)} P_m\left(-(b/\pi)^2 \frac{\partial^2}{\partial y^2}\right) \sin(n\pi y/b) = L_N[\sin(n\pi y/b)]. \tag{22}$$

To get the first, second and third equalities, equations (18), (19) and (21) are used, respectively.

In order to obtain specific boundary conditions, the simplest choice possible for the polynomials P_m is made, namely $P_m(k) = k^m$. Then (18) and (21) become,

$$n = \sum_{m=0}^{N-1} \alpha_m^{(N)} n^{2m}; \qquad n = 1, \ldots, N \tag{23}$$

$$L_N = \sum_{m=0}^{N-1} \alpha_m^{(N)} (-1)^m (b/\pi)^{2m} \frac{\partial^{2m}}{\partial y^{2m}}. \tag{24}$$

The local boundary condition (17) becomes,

$$\frac{\partial u}{\partial x} = \sum_{m=0}^{N-1} \alpha_m^{(N)} (-1)^{m+1} (b/\pi)^{2m-1} \frac{\partial^{2m}}{\partial y^{2m}} u \qquad \text{on } B. \tag{25}$$

This is the desired boundary condition.

It only remains to find the coefficients $\alpha_m^{(N)}$. This is done by solving the system (23), which can be written in a matrix form as,

$$\begin{pmatrix} 1^0 & 1^2 & 1^4 & \cdots & \\ 2^0 & 2^2 & 2^4 & \cdots & \\ \vdots & & & & \\ N^0 & N^2 & N^4 & \cdots & N^{2(N-1)} \end{pmatrix} \begin{Bmatrix} \alpha_0^{(N)} \\ \alpha_1^{(N)} \\ \vdots \\ \alpha_{N-1}^{(N)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ \vdots \\ N \end{Bmatrix} \tag{26}$$

The solutions of this system for $N = 1, 2, 3$ are as follows,

$$N=1: \quad \alpha_0^{(1)} = 1 \tag{27}$$

$$N=2: \quad \alpha_0^{(2)} = 2/3, \quad \alpha_1^{(2)} = 1/3 \tag{28}$$

$$N=3: \quad \alpha_0^{(3)} = 3/5, \quad \alpha_1^{(3)} = 5/12, \quad \alpha_2^{(3)} = -1/60 \tag{29}$$

Thus, the first three local boundary conditions on B are obtained from (25) and (27)–(29),

$$N=1: \quad \frac{\partial u}{\partial x} = -\frac{\pi}{b} u \tag{30}$$

$$N=2: \quad \frac{\partial u}{\partial x} = -\frac{2}{3}\frac{\pi}{b} u + \frac{1}{3}\frac{b}{\pi}\frac{\partial^2 u}{\partial y^2} \tag{31}$$

$$N=3: \quad \frac{\partial u}{\partial x} = -\frac{3}{5}\frac{\pi}{b} u + \frac{5}{12}\frac{b}{\pi}\frac{\partial^2 u}{\partial y^2} + \frac{1}{60}\left(\frac{b}{\pi}\right)^3 \frac{\partial^4 u}{\partial y^4} \tag{32}$$

LOCALIZED BOUNDARY CONDITIONS: CASE 1

Now consider *Case 1*. On B, any solution u consisting of the first N harmonics has the form

$$u(x_0, y) = \sum_{n=0}^{N} A_n \cos(n\pi y/b). \tag{33}$$

As before, (33) is substituted in the exact non-local condition (9), to obtain,

$$\frac{\partial u}{\partial x}(x_0, y) = V_\infty - \frac{\pi}{b} \sum_{n=1}^{\infty} A_n n \cos(n\pi y/b). \tag{34}$$

Now, suppose a linear differential operator L_N is found which satisfies,

$$n \cos(n\pi y/b) = L_N[\cos(n\pi y/b)], \qquad n = 1, \ldots, N. \tag{35}$$

Then (34) gives,

$$\frac{\partial u}{\partial x}(x_0, y) = V_\infty - \frac{\pi}{b} L_N\left[\sum_{n=1}^{\infty} A_n \cos(n\pi y/b)\right], \tag{36}$$

and (33) is used to obtain,

$$\frac{\partial u}{\partial x}(x_0, y) = V_\infty - \frac{\pi}{b} L_N[u(x_0, y) - A_0]. \tag{37}$$

Note the appearance of A_0 in the right side of (37). From (33) it is clear that A_0 is the average of $u(x_0, y)$, and therefore it will give rise to a *non-local* term in the boundary condition (37) if it is retained. There are two ways to make sure that A_0 is not retained in (37) (thus leading to a purely local boundary condition),

(a) If it is known in advance that the average of u along B is zero, then $A_0 = 0$. In some problems where a Neumann boundary condition is imposed along the entire boundary, an additional condition is required in order to render the solution unique. The condition that the average of u vanishes across the width of the channel may serve as such an additional condition.

(b) If the linear operator $L_N[u]$ in (37) involves only derivatives of u but not u itself, then $L_N[A_0] = 0$.

Method (b) will be chosen for dropping out A_0, because it is general. Thus, one cannot use (24) (or (21)) to define L_N, due to the term $m = 0$ which appears there. Instead, (23) and (24) are replaced by,

$$n = \sum_{m=1}^{N} \alpha_m^{(N)} n^{2m}; \qquad n = 1, \ldots, N \tag{38}$$

$$L_N = \sum_{m=1}^{N} \alpha_m^{(N)} (-1)^m (b/\pi)^{2m} \frac{\partial^{2m}}{\partial y^{2m}}, \tag{39}$$

where the sums begin with $m = 1$, unlike in (23) and (24). Then (37) yields,

$$\frac{\partial u}{\partial x} = V_\infty + \sum_{m=1}^{N} \alpha_m^{(N)} (-1)^{m+1} (b/\pi)^{2m-1} \frac{\partial^{2m}}{\partial y^{2m}} u \qquad \text{on } B. \tag{40}$$

This is the desired boundary condition for *Case 1*. Note that the price one has to pay in order to drop out A_0 from (37) is the increase in the order of the derivatives in the local boundary conditions; whereas in *Case 2* the Nth condition involves derivatives up to order $2N-2$, in *Case 1* the Nth condition involves derivatives up to order $2N$. In both cases, the Nth condition is exact for solutions consisting of up to N harmonics.

The system (38) is written in a matrix form as,

$$\begin{pmatrix} 1^2 & 1^4 & 1^6 & \cdots & \\ 2^2 & 2^4 & 2^6 & \cdots & \\ \vdots & & & & \\ N^2 & N^4 & N^6 & \cdots & N^{2N} \end{pmatrix} \begin{Bmatrix} \alpha_1^{(N)} \\ \alpha_2^{(N)} \\ \vdots \\ \alpha_N^{(N)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ \vdots \\ N \end{Bmatrix} \tag{41}$$

The solutions of this system for $N=1, 2$ are as follows,

$$N=1: \qquad \alpha_1^{(1)}=1 \tag{42}$$

$$N=2: \qquad \alpha_1^{(2)}=7/6, \qquad \alpha_2^{(2)}=-1/6 \tag{43}$$

Thus, the first two local boundary conditions on B are obtained from (40), (42) and (43),

$$N=1: \qquad \frac{\partial u}{\partial x}=V_\infty+\frac{b}{\pi}\frac{\partial^2 u}{\partial y^2} \tag{44}$$

$$N=2: \qquad \frac{\partial u}{\partial x}=V_\infty+\frac{7}{6}\frac{b}{\pi}\frac{\partial^2 u}{\partial y^2}+\frac{1}{6}\left(\frac{b}{\pi}\right)^3\frac{\partial^4 u}{\partial y^4} \tag{45}$$

FINITE ELEMENT FORMULATION: NON-LOCAL BOUNDARY CONDITIONS

Now consider the problem in Ω; see *Figure 1*. To fix ideas, suppose the flow is potential throughout $\Omega \cup D$, so that Laplace's equation governs in Ω too. (The formulation may be generalized, however, to more complicated governing equations in Ω.) Let $S=S_g \cup S_h$, where Dirichlet and Neumann conditions are given on S_g and S_h, respectively. Thus,

$$\nabla^2 u=0 \qquad \text{in } \Omega, \tag{46}$$

$$u=g \qquad \text{on } S_g; \qquad \frac{\partial u}{\partial \nu}=h \qquad \text{on } S_h, \tag{47}$$

where $\partial/\partial \nu$ is the normal derivative on S_h. It is assumed that at the two "corner" points $(x_0, 0)$ and (x_0, b) the boundary conditions given on Γ_L and Γ_U are consistent with those given by (47). On B, one of the non-local boundary conditions (9) or (12) is imposed. They both have the form,

$$\frac{\partial u}{\partial x}=C+Mu \qquad \text{on } B, \tag{48}$$

where C is a constant and M is the non-local (DtN) operator defined by (9) or (12).

Define the two sets,

$$V=\{v \mid v \text{ is regular, } v=g \text{ on } S_g\} \tag{49}$$

$$V_0=\{v \mid v \text{ is regular, } v=0 \text{ on } S_g\} \tag{50}$$

The regularity indicated in (49) and (50) is the requirement that $v \in H^1(\Omega)$. (Here H^1 is the 1st-order Sobolev space.) Then the weak form of (46)–(48) is as follows,

Find $u \in V$ such that for all $w \in V_0$ there holds,

$$a(w, u)+b(w, u)=p(w). \tag{51}$$

Here,

$$a(w, u)=\int_\Omega \nabla w \cdot \nabla u \, d\Omega, \tag{52}$$

$$b(w,u)=-\int_B wMu\,\mathrm{d}B, \tag{53}$$

$$p(w)=\int_B wC\,\mathrm{d}B+\int_{S_h} wh\,\mathrm{d}S. \tag{54}$$

After the finite element discretization, this weak form of the problem leads to the following finite element matrix problem,

$$\mathbf{Kd}=\mathbf{F} \tag{55}$$

$$\mathbf{K}=\mathbf{K}^a+\mathbf{K}^b \tag{56}$$

$$\mathbf{K}^a=A_{e=1}^{N_{el}}(\mathbf{k}^a)^e, \qquad \mathbf{F}=A_{e=1}^{N_{el}}\mathbf{f}^e \tag{57}$$

$$(\mathbf{k}^a)^e=[(k^a)^e_{ab}], \qquad \mathbf{f}^e=\{f^e_a\}, \qquad \mathbf{K}^b=[K^b_{AB}] \tag{58}$$

$$(k^a)^e_{ab}=a(N_a,N_b)^e, \qquad f^e_a=p(N_a)^e-\sum_b g^e_b(k^a)^e_{ab} \tag{59}$$

$$K^b_{AB}=b(N_A,N_B) \tag{60}$$

Here **K**, **d** and **F** are the global stiffness matrix, solution vector and load vector, respectively, $A_{e=1}^{N_{el}}$ is the assembly operator, g^e_b is the value of the datum g at node b of element e, N_a is the element shape function associated with local node a, and N_A is the global shape function associated with global node A. The bilinear form $a(\cdot,\cdot)^e$ and the linear form $p(\cdot)^e$ are the element counterparts of the global forms (52) and (54).

As indicated by (56), the global stiffness matrix **K** is the sum of two matrices: $\mathbf{K}^a$, the standard matrix associated with the Laplacian, and $\mathbf{K}^b$, the matrix associated with the boundary condition on B. Note that while $\mathbf{K}^a$ and the load vector **F** are formed on the element level (cf. (59)), as usual in finite element schemes, the matrix $\mathbf{K}^b$ must be formed *on the global level* (cf. (60)), due to the non-local character of the operator M in (53).

FINITE ELEMENT FORMULATION: LOCAL BOUNDARY CONDITIONS

Now consider the problem consisting of (46), (47), and one of the *local* boundary conditions (30)–(32), (44) or (45). All these conditions have the general form,

$$\frac{\partial u}{\partial x}=C+Au+B\frac{\partial^2 u}{\partial y^2}+D\frac{\partial^4 u}{\partial y^4} \qquad \text{on } B. \tag{61}$$

Here C, A, B and D are constants.

The weak form of the problem is similar to the one in the previous section, except for two differences. First, the regularity build in (49) and (50) should be interpreted as the requirement that $v\in H^1(\Omega)$, and if $D\neq 0$ in (61) then in addition the trace of v on B must be in H^2. Second, (53) is replaced by,

$$b(w,u)=-\int_B w\left(Au+B\frac{\partial^2 u}{\partial y^2}+D\frac{\partial^4 u}{\partial y^4}\right)\mathrm{d}B. \tag{62}$$

The bilinear form $b(\cdot,\cdot)$ is not written in (62) in a symmetric way. It may be partly symmetrized by applying to it integration by parts,

$$b(w,u)=-\int_B wAu\,\mathrm{d}B+\int_B \frac{\partial w}{\partial y}B\frac{\partial u}{\partial y}\mathrm{d}B-\int_B \frac{\partial^2 w}{\partial y^2}D\frac{\partial^2 u}{\partial y^2}\mathrm{d}B$$
$$-[wB(\partial u/\partial y)]^*-[wD(\partial^3 u/\partial y^3)]^*+[(\partial w/\partial y)D(\partial^2 u/\partial y^2)]^*. \tag{63}$$

Here,

$$[f]^* \equiv f(x_0, b) - f(x_0, 0). \tag{64}$$

The three first terms in (63) are symmetric. The last three terms in (63), of the form $[\cdot]^*$, are "corner" terms, which are *non-symmetric*. However, the first two of these terms vanish in both *Case 1* and *Case 2*. To see this, note that in *Case 2*, $w=0$, as well as $u=0$, at the corner points. (Recall that w must satisfy the homogeneous counterpart of the essential boundary conditions satisfied by u; cf. (49) and (50).) In *Case 1*, $\partial u/\partial y=0$ and also $\partial^3 u/\partial y^3=0$ at the corner points. The latter relation may be obtained by differentiating Laplace's equation (1) with respect to y, and noting that (2) implies $\partial^3 u/\partial y\, \partial x^2=0$ on Γ_U and Γ_L.

The last term in (63) also vanishes in *Case 2*, since $\partial^2 u/\partial y^2=0$ at the corner points. This again is obtained by differentiating (1) with respect to y, and noting that $\partial^2 u/\partial x^2=0$ on Γ_U and Γ_L, from (4). On the other hand, in *Case 1* the last term in (63) does not automatically vanish, unless $D=0$. However, it is possible to slightly modify the definitions of the sets V and V_0 in (49) and (50) so as to make this term zero too, without loss of generality. To this end, the extra requirement that $\partial u/\partial y=\partial w/\partial y=0$ at the corner points, is built into the definitions of V and V_0 (in *Case 1* only). The requirement will be enforced later in the finite element discretization process, by including $\partial u/\partial y$ at the two corners as degrees of freedom, and fixing their values to zero.

Thus, the bilinear form $b(\cdot,\cdot)$ in (63) is reduced to the symmetric form,

$$b(w, u) = -\int_B wAu\, \mathrm{d}B + \int_B \frac{\partial w}{\partial y} B \frac{\partial u}{\partial y}\, \mathrm{d}B - \int_B \frac{\partial^2 w}{\partial y^2} D \frac{\partial^2 u}{\partial y^2}\, \mathrm{d}B. \tag{65}$$

Similarly, when even higher-order boundary conditions are used on B, additional non-symmetric corner terms appear in the finite element formulation, as in (63). However, some of these terms vanish from the outset, and it is usually possible to pose appropriate restrictions on the function space so as to render the other corner terms zero too. Even without such extra restrictions, it can be verified experimentally that these terms are very small in magnitude with respect to the symmetric terms, and so can be neglected from a computational viewpoint. Note that this issue *does not arise in exterior problems*[19], where the artificial boundary B is *closed*, and therefore the integration by parts used to derive (63) does not produce "corner" terms.

The finite element matrix problem is similar to (55)–(60), except that the formation of $\mathbf{K}^b$ may be performed now *on the element level*,

$$\mathbf{K}^b = A_{e=1}^{N_{el}} (\mathbf{k}^b)^e, \qquad (\mathbf{k}^b)^e = [(k^b)^e_{ab}] \tag{66}$$

$$(k^b)^e_{ab} = b(N_a, N_b)^e. \tag{67}$$

Here, $b(\cdot,\cdot)^e$ is the element counterpart of the bilinear form (65).

It is important to note that if $D \neq 0$ in (65), then *special finite elements* with C^1 continuity along B must be used in a single layer near B. The reason is that if $D \neq 0$ then a functions u^h in the finite element space must be in $H^1(\Omega)$ and its trace on B must be in H^2. Since functions in the finite element space are piecewise-polynomials, these requirements imply that u^h is in $C^0(\Omega)$ and in $C^1(B)$. This means that in the case where second-order derivatives appear in the weak form of the problem, as in (65), the finite elements used in the layer near B must possess C^1 continuity along B and C^0 continuity elsewhere. Such a $C^1(B)$ element was developed by Givoli and Keller[21], in another context. Here, it is used in conjunction with local boundary conditions of the form (61).

Properties of the stiffness matrix

In this section the symmetry, positivity and sparseness properties of the stiffness matrix $\mathbf{K}$ are discussed in the cases where non-local and local boundary conditions are used on B. Since the standard stiffness matrix $\mathbf{K}^a$ (cf. (56)–(59)) is known to be symmetric, positive definite and banded, it must be checked whether $\mathbf{K}^b$ also possesses these properties.

If the non-local condition (9) or (12) is used on B, it can easily be verified that the matrix $\mathbf{K}^b$, whose entries are given in (60), is symmetric. This follows directly from (9) and (12), and can also be proved in a general setting[19]. In the case where a local boundary condition is used on B, it has already been observed that $\mathbf{K}^b$ is symmetric.

In order to guarantee the positive definiteness of $\mathbf{K}$, the matrix $\mathbf{K}^b$ must be positive semi-definite. If the non-local condition (9) or (12) is used on B, this can indeed be shown to hold. The positivity of $\mathbf{K}^b$ follows from the fact that $b(v, v)$ in (53) is non-negative for any continuous function v defined on B. This in turn follows directly from (9) and (12).

Similarly, if a local condition of the form (61) is used on B, then a necessary condition for $\mathbf{K}^b$ to be positive semi-definite is that $b(v, v)$ in (65) is non-negative for any v. From (65), it is clear that this condition is satisfied if the constants A, B and D satisfy $A \leqslant 0$, $B \geqslant 0$ and $D \leqslant 0$. By comparing the boundary conditions (30)–(32), (44) and (45) to the general form (61), it follows that indeed $A \leqslant 0$ and $B \geqslant 0$ for all five boundary conditions. The low-order boundary conditions (44), (30) and (31) also satisfy $D = 0$, and thus lead to positive definite stiffness matrices. However, in the high-order boundary conditions (45) and (32), $D > 0$. This means that the last term in (65) may render the matrix $\mathbf{K}^b$, and thus maybe also $\mathbf{K}$, indefinite.

Numerical investigation reveals that indeed $\mathbf{K}$ typically becomes indefinite when (45) or (32) are used, with negative values in and around the diagonal entries corresponding to degrees of freedom on B. In some cases, the stiffness matrix becomes ill-conditioned, and the numerical results are then sensitive to small perturbations. In order to overcome this sensitivity and to effectively reduce the condition number, one must use special stabilizing procedures while solving the linear system (55).

When a local condition of the form (61) is used on B, the stiffness matrix remains sparse as in a standard finite element scheme, since all the operations are performed on the element level. On the other hand, it may be expected that when the non-local boundary condition (9) or (12) is used on B, the bandedness of $\mathbf{K}$ may be spoiled. In fact, this is not the case. It can easily be verified that the non-zero block in $\mathbf{K}^b$ corresponding to the degrees of freedom on B is entirely contained within the skyline of the matrix $\mathbf{K}^a$. This is demonstrated for wave guides in Givoli[1].

NUMERICAL EXPERIMENTS

To test the accuracy of the different boundary conditions, two model problems are considered, which involve a uniform semi-infinite strip with width $b = 5$, and whose exact solutions are known. The first problem corresponds to *Case 1*, so that (2) and (3) hold. The uniform velocity at infinite is $V_\infty = 1$. The boundary condition $\partial u/\partial y = 0$ is imposed along the upper and lower boundaries, while the boundary condition imposed on the left boundary ($x = 0$) is,

$$u(0, y) = \sum_{n=1}^{2} U_n \cos(n\pi y/b), \tag{68}$$

where $U_1 = 100$ and $U_2 = 1000$. The exact solution to this problem is,

$$u(x, y) = V_\infty x + \sum_{n=1}^{2} U_n \cos(n\pi y/b) \exp(-n\pi x/b). \tag{69}$$

Thus, the exact solution consists of the first two harmonics.

The artificial boundary B is set at $x = x_0 = 5$. (See *Figure 1*.) Thus, the computational domain Ω is the rectangle $0 \leqslant x \leqslant 5$, $0 \leqslant y \leqslant 5$. The finite element mesh consists of 59×60 identical bilinear square elements, and 60 special $C^1(B)$ finite elements[21] in the layer adjacent to B. On B, five different boundary conditions are imposed: the uniform-flow condition $\partial u/\partial x = V_\infty$, the two non-local conditions obtained from (9) by truncating the infinite sum after the first term ($N = 1$) and after the first two terms ($N = 2$), and the two local conditions (44) ($N = 1$) and (45) ($N = 2$).

Figure 2 shows the numerical solutions for u along B, obtained by using these five boundary conditions, as well as the exact solution (69). It is apparent that both the non-local and local conditions corresponding to $N=2$ are in excellent agreement with the exact solution. The non-local and local conditions corresponding to $N=1$ yield less accurate results, with the local condition being significantly better. The uniform-flow condition $\partial u/\partial x = V_\infty$ performs very poorly.

Similar results are obtained for the normal derivative $\partial u/\partial x$ on B, shown in *Figure 3*, and for the tangential derivative $\partial u/\partial y$ on B, shown in *Figure 4*. In *Figure 3*, the normal derivative along B corresponding to the boundary condition $\partial u/\partial x=1$ deviates from the value 1 due to finite

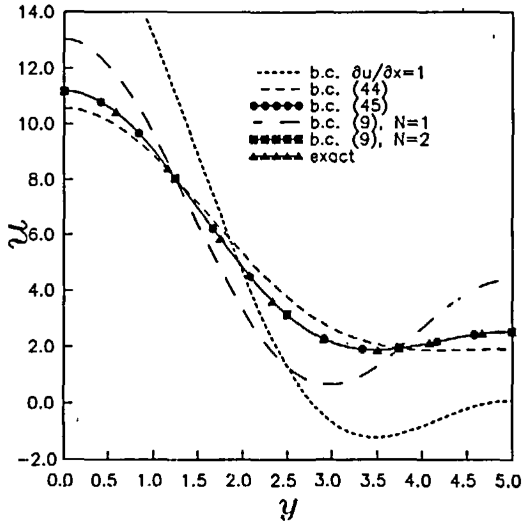

Figure 2 *Case 1*, test problem: results for u on B, using different local and non-local boundary conditions

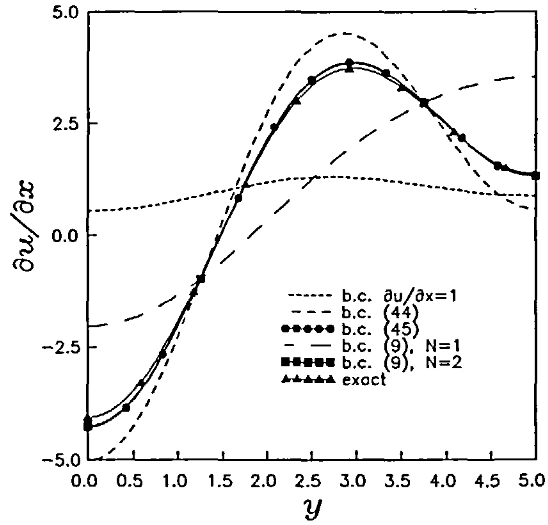

Figure 3 *Case 1*, test problem: results for the normal derivative on B, using different local and non-local boundary conditions

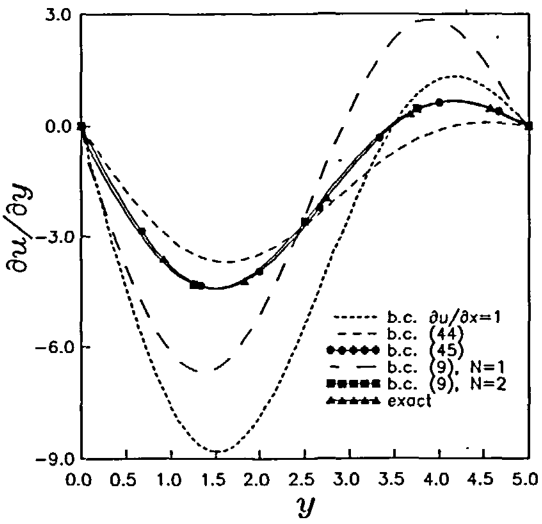

Figure 4 *Case 1*, test problem: results for the tangential derivative on B, using different local and non-local boundary conditions

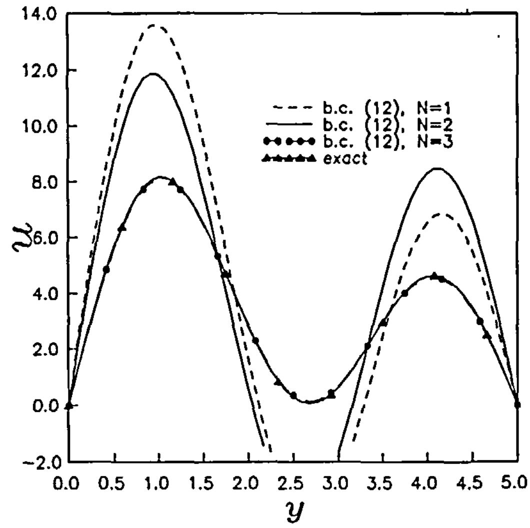

Figure 5 *Case 2*, test problem: results for u on B, using different non-local boundary conditions

element discretization error. In *Figure 4*, note that the finite element approximation of the tangential derivative is continuous along B, owing to the use of the $C^1(B)$ elements.

The second model problem corresponds to *Case 2*, so that (4) and (5) hold. The boundary condition $u=0$ is imposed along the upper and lower boundaries, while on the left boundary ($x=0$) the function u is prescribed as,

$$u(0, y)=\sum_{n=1}^{3} U_n \sin(n\pi y/b), \tag{70}$$

where $U_1=100$, $U_2=1000$ and $U_3=50{,}000$. The exact solution to this problem is,

$$u(x, y)=\sum_{n=1}^{3} U_n \sin(n\pi y/b) \exp(-n\pi x/b), \tag{71}$$

which consists of the first three harmonics. The same computational parameters are used as in the previous case. On B, seven different boundary conditions are imposed: the condition $\partial u/\partial x=0$ (which is correct at infinity but only approximate on B), the three non-local conditions obtained from (12) by truncating the infinite sum after the first term ($N=1$), the first two terms ($N=2$) and the first three terms ($N=3$), and the three local conditions (30) ($N=1$), (31) ($N=2$) and (32) ($N=3$).

Figure 5 shows the numerical solutions for u along B, obtained by using the *non-local* boundary conditions, and the exact solution (71). In *Figure 6* the results for the *local* boundary conditions are shown. Both non-local and local conditions corresponding to $N=3$ agree very well with the exact solution. The maximal relative errors generated by the local conditions (31) ($N=2$) and (30) ($N=1$), the non-local conditions with $N=2$ and with $N=1$, and the condition $\partial u/\partial x=0$, are 6%, 31%, 46%, 71% and 84% respectively. Note again the superiority of the low-order local conditions over the non-local ones.

In terms of CPU times, the most economic boundary condition is, of course, the condition $\partial u/\partial x=0$. In the present example, the local $N=2$ condition, the non-local $N=3$ condition and the local $N=3$ condition were 1.6%, 2.6% and 5.7% more costly, respectively. Thus, the various

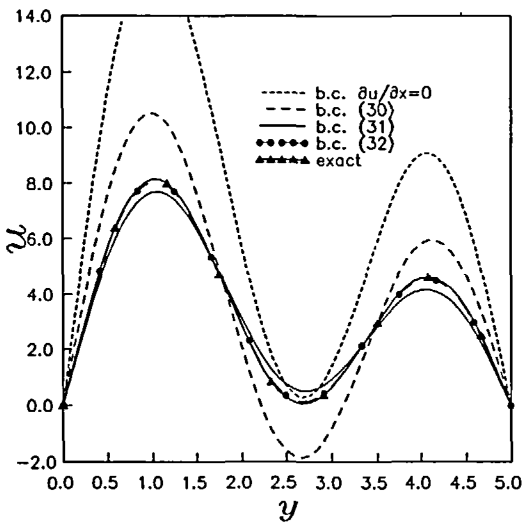

Figure 6 *Case 2*, test problem: results for u on B, using different local boundary conditions

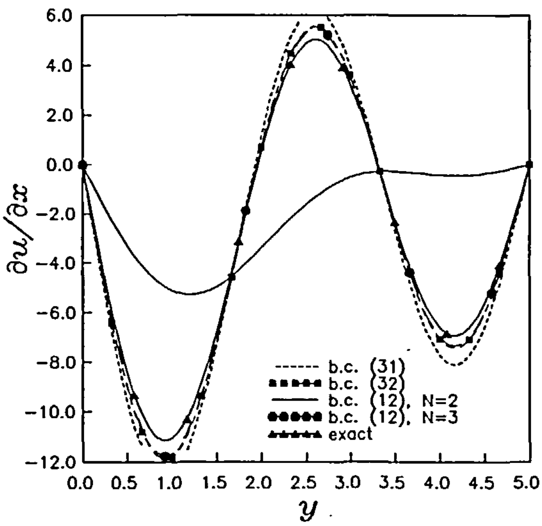

Figure 7 *Case 2*, test problem: results for the normal derivative on B, using different local and non-local boundary conditions

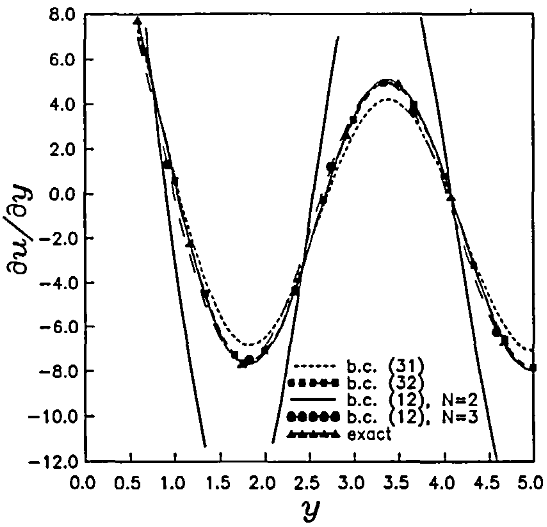

Figure 8 *Case 2*, test problem: results for the tangential derivative on B, using different local and non-local boundary conditions

conditions require about the same computational effort. It should be noted, however, that in the local $N=3$ case, this effort strongly depends upon the number of special C^1 elements used in the mesh.

In *Figures 7* and *8*, the results for the normal and tangential derivatives along B are shown. Four boundary conditions are compared in the figures: the local and non-local conditions corresponding to $N=2$ and $N=3$. The relative accuracy of the different boundary conditions is similar to that observed previously. Again, the non-local condition with $N=2$ is dramatically worse than the local $N=2$ condition.

These experiments, as well as others which are not presented here, lead to the following conclusion: a local boundary condition of the form (25) or (40), of order N, is significantly more accurate than its non-local counterpart (12) or (9) of the same order N (namely truncated after N terms), in resolving the higher modes $M>N$ of the exact solution. The explanation to this phenomenon is that due to the orthogonality property built in the non-local conditions (9) and (12), these conditions completely annihilate all the higher modes. In contrast, *all* the modes contribute to the right hand side of the local conditions (25) and (40). In fact, numerical experiments as well as rigorous analysis[22] show that the local conditions perform very well in resolving sufficiently smooth solutions with many modes. Of course, the finite element mesh must be fine enough to capture the high-order modes.

Finally, the problem of potential flow past a circular cylinder in a channel is considered. The width of the channel is $b=5$ and the radius of the cylinder is $R=2$. Here u is the velocity potential, and a no-penetration boundary condition is applied on the lower and upper boundaries. A uniform flow with velocity $\partial u/\partial x=1$ is enforced at the left boundary ($x=0$), and also at infinity. The center of the cylinder cross section is located at a distance 2.5 from the left and from the lower boundaries. Thus, the problem is symmetric with respect to the axis $y=b/2$. At the point $(0, b/2)$ the arbitrary value $u=0$ is prescribed to render the solution of the problem unique. The artificial boundary B is located at $x=5.1$. The local boundary condition (45) is used on B. The finite element mesh is shown in *Figure 9*. It consists of bilinear quadrilateral elements, except near B where a layer of $C^1(B)$ elements is used.

Figure 10 shows the contour lines of the velocity potential u. It is clear that the flow is far from being uniform at the right (artificial) boundary B of the computational domain Ω. Thus,

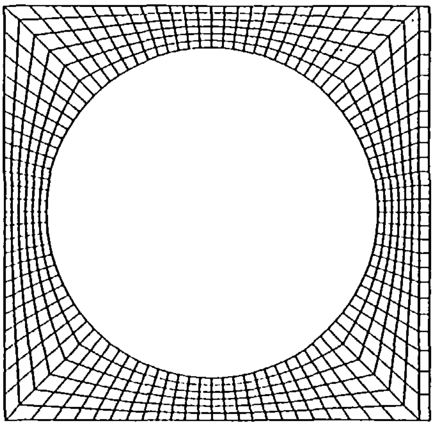

Figure 9 Potential flow past a cylinder in a channel: finite element mesh

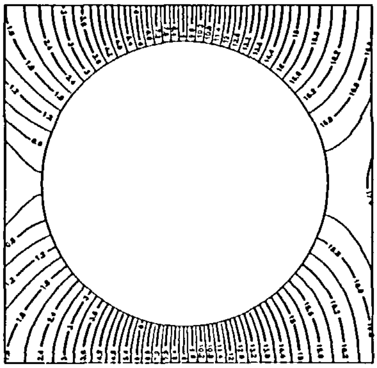

Figure 10 Potential flow past a cylinder in a channel: contour lines of velocity potential

the use of the high-order artificial boundary condition enables the use of a small computational domain. In fact, the computational domain chosen in this example is not much larger than the cylindrical obstacle. To achieve the same accuracy with the standard uniform-flow boundary condition $\partial u/\partial x = V_\infty$ on B, the size of the computational domain must be significantly larger than the size of the obstacle[7].

A similar numerical solution may be obtained with the use of the non-local boundary condition (9) on B. For some other problems, where high modes are paticularly important, the local condition (45) exhibits unstable behavior due to ill-conditioning of the stiffness matrix, as noted previously. In these cases, the non-local condition (9) continues to yield well-behaved solutions.

CONCLUDING REMARKS

Non-local and local boundary conditions were devised for use on an artificial boundary, for two-dimensional flow problems in an infinite channel. These conditions are exact for solutions consisting of a specified number of harmonics. They are based on the assumption that in the uniform tail region D of the channel the flow is potential. Although the finite element scheme used in this paper was based on a similar assumption in the computational domain Ω, the same ideas may be carried over to more general equations in Ω, representing more complicated flows in the region near the obstacle and/or sources. Work on more complicated flows, such as a two-dimensional rotational flow, will be reported in the future. Artificial boundary conditions for differential equations other than Laplace's equation in an infinite channel are analyzed in Reference 22.

In Reference 22 other aspects of the subject are also considered, which were not dealt with here. These include the development of detailed error estimates (and their proofs), which provide the rate of convergence for the finite element scheme using local and non-local boundary conditions, as a function of the discretization parameters and the order N of the boundary condition. In addition the behaviour of the numerical solutions as a function of the position of the artificial boundary is investigated.

The non-local conditions are very robust. They always lead to a symmetric positive-definite stiffness matrix, and in the present case they do not alter the skyline structure of this matrix. Moreover, once they are implemented in the finite element code, one may use them very easily, taking into account any desired number of terms. Their main disadvantage is that they require computations on the global level, which is contrary to the usual architecture of finite element codes. Therefore they are also less amenable for parallelization than local conditions.

The local boundary conditions have the advantage that they are incorporated in a finite element code in the usual manner, i.e. all the calculations are performed on the element level. The low-order localized conditions are simple, but not always sufficiently accurate. However, they are much more accurate than their non-local counterparts *of equal order* in resolving the higher modes in the exact solution. For problems where the first few modes are dominant they should be satisfactory.

The high-order localized conditions are more accurate. However, they require the use of special finite elements in the layer adjacent to B^{21}. In addition, some of them, like the C^1-type boundary conditions (32) and (45), lead to an indefinite stiffness matrix which is typically also ill-conditioned. This sometimes gives rise to instabilities in the numerical solution. It should be remarked that such instability was *not* observed in the numerical examples that were presented here. However, some instability difficulties were encountered in a few other examples.

To overcome this potential instability, one may try to apply special algebraic stabilizing procedures directly to the (indefinite) finite element system (55), or one may resort to a special finite element formulation which has the effect of stabilizing the numerical solution, like Galerkin Least Squares (GLS). Harari and Hughes[23] show, in the context of the reduced wave equation, that GLS can help in stabilizing an otherwise ill-conditioned formulation. Another and probably better option is to use a still higher-order local condition which is known to be stable. Reference 22 shows that C^2-type local boundary conditions are indeed stable, as are all the C^N-type conditions with N even.

Thus, the superiority of one type of artificial boundary conditions over the other seems to be problem and code dependent. A more detailed characterization of the advantages and disadvantages of non-local versus local boundary conditions, mainly in the context of acoustic waves, is given elsewhere[22].

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